An Analytical Determination of Certain Line Groups in the Plane

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Mathematics

An Analytical Determination of
Certain Lie Groups in the Plane

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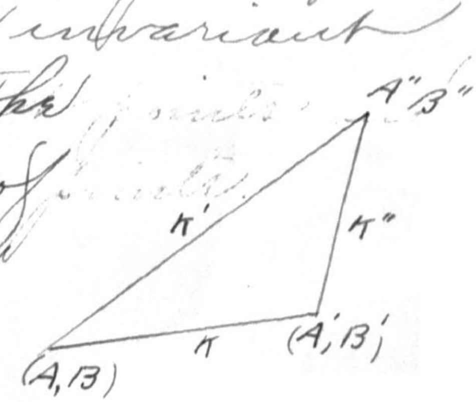
The most general collineation in the plane is given by the following equations

$$x_1 = \frac{ax+by+c}{a''x+b''y+c''}, \quad y_1 = \frac{a'x+b'y+c'}{a''x+b''y+c''} \qquad (I)$$

These equations contain eight independent arbitrary constants and therefore there are possible ∞^8 such collineations in the plane. It has been shown that these ∞^8 collineations form a group. This group is called the general projective group of the plane and is an eight parameter group, G_8.

Each collineation of the group mentioned above leaves invariant a triangle. Let us take the coordinates of the vertices of this invariant triangle to be (A,B); (A',B'), (A'',B''). There will be one dimensional collineations along the three invariant lines. Let the cross ratios of these one dimensional collineations be K, K', K''. It



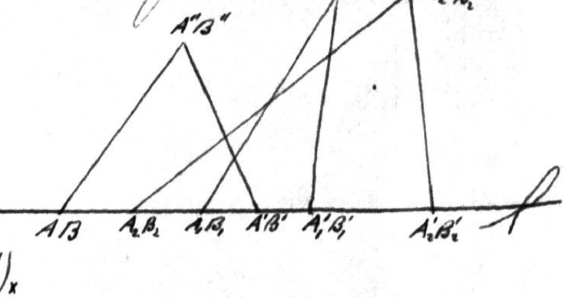
has been shown that these three cross ratios are not independent but that their product taken in the same direction around the triangle is unity. Thus any two of the three determines the third.

Prof Newson has shown that the equation of the general collineation in the plane may be expressed in terms of the coordinates of the vertices of the invariant triangle of the collineation and of the two independent cross ratios of the one dimensional transformations along the sides of this invariant triangle. Using determinants we may write it in the following simple concise form

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ A' & B' & 1 & KA' \\ A'' & B'' & 1 & K'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ A' & B' & 1 & KB' \\ A'' & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad (II)$$

The eight parameters of these equations are seen to be $A, A', A'', B, B', B'', K, K'$ or the six coordinates of the three invariant points and the two independent cross-ratios of the one dimensional transformations along the sides of the invariant triangle. By the use of equations (II) the subgroups of the eight parameter projective group may be quite easily developed. To determine certain of these subgroups by the analytic process is the purpose of this paper.

The first system of collineations we wish to investigate is that system leaving invariant a certain line l in the plane.



If we take l for the X axis the coordinates of the vertices of the invariant triangle are $(A, 0)$, $(A', 0)$ (A'', B'').
We can get the equations of a collineation of this system by putting $B = 0$, $B' = 0$ in equations (II). If we make this substitution in (II) we get the following equations.

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & 0 & 1 & A \\ A' & 0 & 1 & KA' \\ A'' & 0 & 1 & K'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & 0 & 1 & 0 \\ A' & 0 & 1 & 0 \\ A'' & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad (III)$$

A second collineation leaving invariant another triangle one of whose sides is l will be of the same form and may be written

$$x_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ A_1 & 0 & 1 & A_1 \\ A_1' & 0 & 1 & K_1A_1' \\ A_1'' & 0 & 1 & K_1'A_1'' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ A_1 & 0 & 1 & 1 \\ A_1' & 0 & 1 & K_1 \\ A_1'' & B_1'' & 1 & K_1' \end{vmatrix}} \qquad y_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ A_1 & 0 & 1 & 0 \\ A_1' & 0 & 1 & 0 \\ A_1'' & B_1'' & 1 & K_1'B_1'' \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ A_1 & 0 & 1 & 1 \\ A_1' & 0 & 1 & K_1 \\ A_1'' & B_1'' & 1 & K_1' \end{vmatrix}} \qquad (IV)$$

If we expand the determinants in (III) we get the following equations

$$x_1 = \frac{B''(KA'-A)x - [K'A''(A-A') + KA'(A''-A) + A(A'-A'')]y + AA'B''(1-K)}{B''(K-1)x - [K'(A-A') + K(A''-A) + (A'-A'')]y + B''(A'-AK)}$$

$$y_1 = \frac{K'B''(A'-A)y}{B''(K-1)x - [K'(A-A') + K(A''-A) + (A'-A'')]y + B''(A'-AK)} \qquad \text{(V)}$$

In the same way if we expand equations (IV) we get

$$x_2 = \frac{B_1''(K_1A_1'-A_1)x_1 - [K_1'A_1''(A_1-A_1') + K_1A_1'(A_1''-A_1) + A_1(A_1'-A_1'']y_1 + A_1A_1'B_1''(1-K_1)}{B_1''(K_1-1)x_1 - [K_1'(A_1-A_1') + K_1(A_1''-A_1) + (A_1'-A_1'')]y_1 + B_1''(A_1'-A_1K_1)}$$

$$y_2 = \frac{K_1'B_1''(A_1'-A_1)y_1}{B_1''(K_1-1)x_1 - [K_1'(A_1-A_1') + K_1(A_1''-A_1) + (A_1'-A_1'')]y_1 + B_1''(A_1'-A_1K_1)} \qquad \text{(VI)}$$

Equations (V) are the equations of a collineation leaving l invariant. Equations (VI) are likewise the equations of a collineation leaving l invariant. We wish to ascertain if the aufeinanderfolge (resultant) of collineations (V) and (VI) also leaves l invariant. If this is the case the aufeinanderfolge must be of the form,

$$x_2 = \frac{B_2''(K_2A_2'-A_2)x - [K_2'A_2''(A_2-A_2') + K_2A_2'(A_2''-A_2) + A_2(A_2'-A_2'')]y + A_2A_2'B_2''(1-K_2)}{B_2''(K_2-1)x - [K_2'(A_2-A_2') + K_2(A_2''-A_2) + (A_2'-A_2'']y + B_2''(A_2'-A_2K_2)}$$

$$y_2 = \frac{K_2'B_2''(A_2'-A_2)y}{B_2''(K_2-1)x - [K_2'(A_2-A_2') + K_2(A_2''-A_2) + (A_2'-A_2'']y + B_2''(A_2'-A_2K_2)} \qquad \text{(VII)}$$

where (A_2, B_2), (A_2', B_2'), (A_2'', B_2'') are the vertices of the triangle left invariant by the aufeinanderfolge and K_2, K_2' are the two independent cross ratios along the sides of this same invariant triangle. B_2 and B_2' do not appear in (VII) since they should be zero if the line l is to be left invariant. We shall wish to use (VII) where both fractions are divided through by the constant in the denominator. After such division (VII) appears as follows.

$$x_2 = \frac{\dfrac{K_2 A_2' - A_2}{A_2' - A_2 K_2}x - \dfrac{K_2 A_2''(A_2 - A_2') + K_2 A_2'(A_2'' - A_2) + A_2(A_2' - A_2'')}{B_2''(A_2' - A_2 K_2)}y + \dfrac{A_2 A_2'(1 - K_2)}{(A_2' - A_2 K_2)}}{\dfrac{K_2 - 1}{A_2' - A_2 K_2}x - \dfrac{K_2'(A_2 - A_2') + K_2(A_2'' - A_2) + (A_2' - A_2'')}{B_2''(A_2' - A_2 K_2)}y + 1}$$

$$y_2 = \frac{\dfrac{K_2'(A_2' - A_2)}{A_2' - A_2 K_2}y}{\dfrac{K_2 - 1}{A_2' - A_2 K_2}x - \dfrac{K_2'(A_2 - A_2') + K_2(A_2'' - A_2) + (A_2' - A_2'')}{B_2''(A_2' - A_2 K_2)}y + 1} \qquad \text{(VIII)}$$

We can get the aufeinanderfolge of (V) and (VI) by substituting in (VI) the values of x_1 and y_1 in (V). If we do this we get the following equations.

$$x_2 = \frac{B''B_1''[(K_1A_1'-A_1)(KA'-A)+AA'(1-K_1)(K-1)]x - \{B_1''(K_1A_1'-A_1)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')]+A_1A_1'B''(1-K_1)[K'(A-A')+K(A''-A)+(A'-A'')]+K'B''(A'-A)[K_1'A_1''(A_1-A_1')+K_1A_1'(A_1''-A_1)+A_1(A_1'-A_1'')]\}y+B''B_1''[AA'(K_1A_1'-A_1)(1-K)+A_1A_1'(A'-AK)(1-K_1)]}{B''B_1''[(K_1-1)(KA'-A)+(K-1)(A_1'-A_1K_1)]x - \{B_1''(K_1-1)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')]+B_1''(A_1'-A_1K_1)[K'(A-A')+K(A''-A)+(A'-A'')]+K'B''(A'-A)[K'(A_1-A_1')+K_1(A_1''-A_1)+(A_1'-A_1'')]\}y+B''B_1''[(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)]}$$

(IX)

$$y_2 = \frac{K'K_1'B''B_1''(A'-A)(A_1'-A_1)y}{B''B_1''[(K_1-1)(KA'-A)+(K-1)(A_1'-A_1K_1)]x - \{B_1''(K_1-1)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')]+B_1''(A_1'-A_1K_1)[K'(A-A')+K(A''-A)+(A'-A'')]+K'B''(A'-A)[K'(A_1-A_1')+K_1(A_1''-A_1)+(A_1'-A_1'')]\}y+B''B_1''[(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)]}$$

If we divide both numerator and denominator of each fraction by the constant term in the denominator we get

$$x_2 = \frac{\dfrac{(K_1A_1'-A_1)(KA'-A)+AA'(1-K_1)(K-1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}x - \dfrac{B_1''(K_1A_1'-A_1)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')]+A_1A_1'B''(1-K_1)[K'(A-A')+K(A''-A)+(A'-A'')]+K'B''(A'-A)[K_1'A_1''(A_1-A_1')+K_1A_1'(A_1''-A_1)+A_1(A_1'-A_1'')]}{B''B_1''[(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)]}y + \dfrac{AA'(K_1A_1'-A_1)(1-K)+A_1A_1'(A'-AK)(1-K_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}}{\dfrac{(K_1-1)(KA'-A)+(K-1)(A_1'-A_1K_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}x - \dfrac{B_1''(K_1-1)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')]+B_1''(A_1'-A_1K_1)[K'(A-A')+K(A''-A)+(A'-A'')]+K'B''(A'-A)[K'(A_1-A_1')+K_1(A_1''-A_1)+(A_1'-A_1'')]}{B''B_1''[(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)]}y + 1}$$

(X)

$$y_2 = \frac{\dfrac{K'K_1'(A'-A)(A_1'-A_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}y}{\dfrac{(K_1-1)(KA'-A)+(K-1)(A_1'-A_1K_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}x - \dfrac{B_1''(K_1-1)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')]+B_1''(A_1'-A_1K_1)[K'(A-A')+K(A''-A)+(A'-A'')]+K'B''(A'-A)[K'(A_1-A_1')+K_1(A_1''-A_1)+(A_1'-A_1'')]}{B''B_1''[(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)]}y + 1}$$

(X) is of same form as (VIII) and if it is to be the same collineation the coefficients of X must be respectively equal to those of VIII. Equating coefficients we have the six equations that follow.

$$(1)\qquad \frac{K_2A_2'-A_2}{A_2'-A_2K_2} = \frac{(K_1A_1'-A_1)(KA'-A)+AA'(1-K_1)(K-1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

$$(2)\qquad \frac{K_2'A_2''(A_2-A_2')+K_2A_2'(A_2''-A_2)+A_2(A_2'-A_2'')}{B_2''(A_2'-A_2K_2)} = \frac{B_1''(K_1A_1'-A_1)\left[K'A''(A-A')+KA'(A''-A)+A(A'-A'')\right]+A_1A_1'B_1''(1-K_1)\left[K'(A-A')+K(A''-A)+(A'-A'')\right]+K'B''(A'-A)\left[K_1'A_1''(A_1-A_1')+K_1A_1'(A_1''-A_1)+A_1(A_1'-A_1'')\right]}{B''B_1''\left[(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)\right]}$$

$$(3)\qquad \frac{A_2A_2'(1-K_2)}{A_2'-A_2K_2} = \frac{AA'(K_1A_1'-A_1)(1-K)+A_1A_1'(A'-AK)(1-K_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

(XI)

$$(4)\qquad \frac{K_2-1}{A_2'-A_2K_2} = \frac{(K_1-1)(KA'-A)+(K-1)(A_1'-A_1K_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

$$(5)\qquad \frac{K_2'(A_2-A_2')+K_2(A_2''-A_2)+(A_2'-A_2'')}{B_2''(A_2'-A_2K_2)} = \frac{B_1''(K_1-1)\left[K'A''(A-A')+KA'(A''-A)+A(A'-A'')\right]+B_1''(A_1'-A_1K_1)\left[K'(A-A')+K(A''-A)+(A'-A'')\right]+K'B''(A'-A)\left[K_1'(A_1-A_1')+K_1(A_1''-A_1)+(A_1'-A_1'')\right]}{B''B_1''\left[(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)\right]}$$

$$(6)\qquad \frac{K_2'(A_2'-A_2)}{(A_2'-A_2K_2)} = \frac{K'K_1'(A'-A)(A_1'-A_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

Equations (IX) are of just the same form as equations (V.) and (VI) and the six equations (XI) uniquely determine A_2, A_2', A_2'', B_2'', the vertices of a new invariant triangle one of whose sides is l, and κ_2, κ_2' the cross ratios of the one dimensional transformations along its sides. Thus we see that the aufeinanderfolge of two collineations leaving l invariant is a collineation of the same form which leaves l invariant.

We can get the inverse of a collineation of this system by solving equations V. for x and y in terms of x_1 and y_1. We shall give the inverse with both fractions divided through by the constant term of the denominator. If the inverse is to belong to the system it must be of the form

$$x_2 = \frac{\dfrac{K_3A_3'-A_3}{A_3'-A_3K_3}x_1 - \dfrac{K_3A_3''(A_3-A_3')+K_3A_3'(A_3''-A_3)+A_3(A_3'-A_3'')}{B_3''(A_3'-A_3K_3)}y_1 + \dfrac{A_3A_3'(1-K_3)}{A_3'-A_3K_3}}{\dfrac{K_3-1}{A_3'-A_3K_3}x_1 - \dfrac{K_3'(A_3-A_3')+K_3(A_3''-A_3)+(A_3'-A_3'')}{B_3''(A_3'-A_3K_3)}y_1 + 1}$$

$$y = \frac{\dfrac{K_3'(A_3'-A_3)}{A_3'-A_3K_3}y_1}{\dfrac{K_3-1}{A_3'-A_3K_3}x_1 - \dfrac{K_3'(A_3-A_3')+K_3(A_3''-A_3)+(A_3'-A_2'')}{B_3''(A_3'-A_3K_3)}y_1 + 1} \tag{XII}$$

where A_3, A_3', A_3'', B_3'' are the coordinates of the vertices of a new invariant triangle with l for a side and K_3, K_3' are the crossratios along its sides. The inverse of (V) obtained as suggested above is

$$x = \frac{\dfrac{A'-AK}{KA'-A}x_1 - \dfrac{[K'(A-A')+K(A''-A)+(A'-A'')][AA'(1-K)] - (A'-AK)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')]}{B''(KA'-A)K'(A'-A)}y_1 - \dfrac{AA'(1-K)}{KA'-A}}{\dfrac{1-K}{KA'-A}x_1 - \dfrac{(K-1)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')] - (KA'-A)[K'(A-A')+K(A''-A)+(A'-A'')]}{B''(KA'-A)K'(A'-A)}y_1 + 1}$$

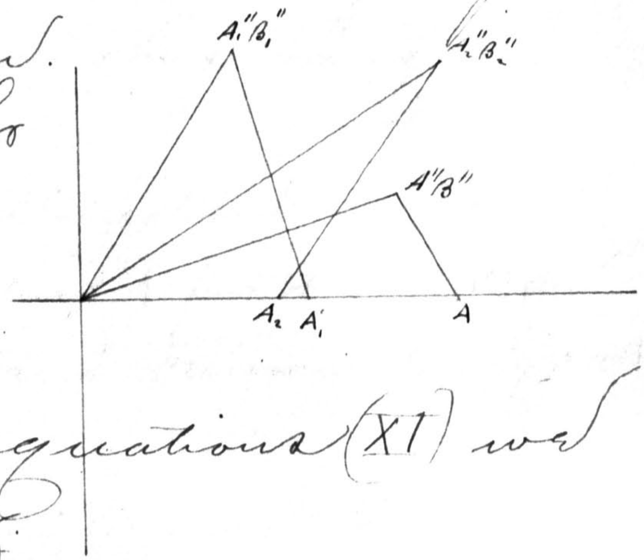
$$y = \frac{\dfrac{(A'-AK)(KA'-A)-(K-1)AA'(1-K)}{(KA'-A)K'(A'-A)}y_1}{\dfrac{1-K}{KA'-A}x_1 - \dfrac{(K-1)[K'A''(A-A')+KA'(A''-A)+A(A'-A'')] - (KA'-A)[K'(A-A')+K(A''-A)+(A'-A'')]}{B''(KA'-A)(K')(A'-A)}y_1 + 1} \tag{XIII}$$

(XIII) is of the same form as (XII) and if we equate coefficients we have six equations which determine uniquely $A_3\ A_3'\ A_3''\ B_3''\ H_3\ K_3'$. Thus the inverse of every collineation in the system under discussion is also in the system. The system contains the identical transformation since the aufeinanderfolge of any collineation and its inverse is an identical collineation. Thus the system possesses all the group properties.

∴ There is a six parameter group leaving invariant a given line in the plane. The parameters are four of the coordinates of the vertices of the invariant triangle and the two independent crossratios of the onedimensional transformations along the sides of the invariant triangle. This is the group $G_6(l)$.

Let us next investigate the system of collineations which leaves invariant a line and a point there on. We will as before take the invariant line l as the axis of x

and in addition take the invariant point there on as the origin. We can bring about this condition of affairs by putting $A_2 = A_1 = A = 0$. If we make these substitutions in equations (XI) we have after reducing



(1) $K_2 = K K_1$

(2) $\dfrac{A_2''(K_2'-K_2)}{B_2''} = \dfrac{K_1 A''(K'-K)}{B''} + \dfrac{K' A_1''(K_1'-K_1)}{B_1''}$

(3) $0 = 0$ an identity (XIV)

(4) $\dfrac{K_2-1}{A_2'} = \dfrac{K(K_1-1)}{A_1'} + \dfrac{K-1}{A'}$

(5) $\dfrac{K_2'-1}{B_2''} - \dfrac{(K_2-1)A_2''}{A_2' B_2''} = \dfrac{(K_1-1)(K'-K)A''}{A_1'' B''} + \dfrac{K'(K_1'-1)}{B_1''} - \dfrac{K' A_1''(K-1)}{A_1' B_1''} + \dfrac{K'-1}{B''} - \dfrac{A''(K-1)}{A' B''}$

(6) $K_2' = K' K_1'$

Thus we see one of the conditions disappear and we have five conditions to determine the five parameters of the transformation. The five parameters are thus uniquely determined×

The equations of a collineation in this system may be written [put $A=0$ in (V) and divide both numerator and denominator by $A'B''$]

$$x_1 = \frac{Kx + \frac{A''(K'-K)}{B''}y}{\frac{K-1}{A'}x + \left(\frac{K'-1}{B''} - \frac{(K-1)A''}{A'B''}\right)y + 1}.$$

$$y_1 = \frac{K'y}{\frac{K-1}{A'}x + \left(\frac{K'-1}{B''} - \frac{(K-1)A''}{A'B''}\right)y + 1} \qquad \text{(XV)}$$

A second collineation may be written

$$x_2 = \frac{K_1x_1 + \frac{A_1''(K_1'-K_1)}{B_1''}y_1}{\frac{K_1-1}{A_1'}x_1 + \left(\frac{K_1'-1}{B_1''} - \frac{(K_1-1)A_1''}{A_1'B_1''}\right)y_1 + 1}$$

$$y_2 = \frac{K_1'y}{\frac{K_1-1}{A_1'}x_1 + \left(\frac{K_1'-1}{B_1''} - \frac{(K_1-1)(A_1'')}{A_1'B_1''}\right)y_1 + 1} \qquad \text{(XVI)}$$

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The aufeinander folge may be written

$$X_2 = \frac{KK_1X + \left(\frac{K_1A''(K'-K)}{B''} + \frac{K'A_1''(K_1'-K_1)}{B_1''}\right)y}{\left(\frac{K(K_1-1)}{A_1'} + \frac{K-1}{A'}\right)X + \left(\frac{(K_1-1)(K'-K)A''}{A_1''B''} + \frac{K'(K_1'-1)}{B_1''} - \frac{K'A_1''(K-1)}{A_1'B_1''} + \frac{K'-1}{B''} - \frac{A''(K-1)}{A'B''}\right)y + 1}$$

$$y_2 = \frac{K'K_1'y}{\left(\frac{K(K_1-1)}{A_1'} + \frac{K-1}{A'}\right)X + \left(\frac{(K_1-1)(K'-K)A''}{A_1''B''} + \frac{K'(K_1'-1)}{B_1''} - \frac{K'A_1''(K-1)}{A_1'B_1''} + \frac{K'-1}{B''} - \frac{A''(K-1)}{A'B''}\right)y + 1} \qquad \underline{(XVII)}$$

and is of the same form.

The inverse may be obtained by solving (XV) for x and y in terms of x_1 and y_1 or more easily by putting $A=0$ in $\underline{XIII}$. In either way we get,

$$x = \frac{\frac{1}{K}x_1 + \frac{A''(K-K')}{B''KK'}y_1}{\frac{1-K}{KA'}x_1 - \frac{K'A''(1-K)+KA'(K'-1)}{KK'A'B''}y_1 + 1}$$

$$\underline{(XVIII)}$$

$$y = \frac{\frac{1}{K'}y_1}{\frac{1-K}{KA'}x_1 - \frac{K'A''(1-K)+KA'(K'-1)}{KK'A'B''}y_1 + 1}$$

This is just the same form as (XV) and will leave invariant the given line and the given point.

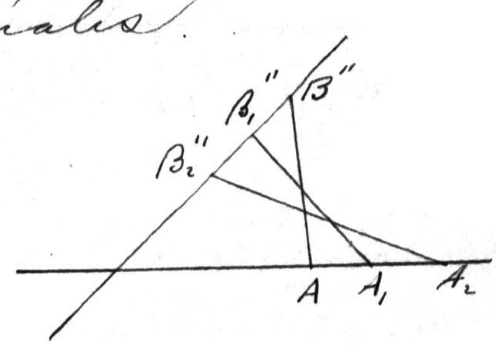
Thus the system under discussion possesses the fundamental group properties.

∴ There is a five parameter group leaving invariant a given line and a given point thereon. The five parameters of the group are B', A_2'', B_2'' K, K'. This is the group G_5-(Al).

The equations of a transformation of this group may be written in determinant form

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & KA' \\ A'' & B'' & 1 & K'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & 0 \\ A'' & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ A'' & B'' & 1 & K' \end{vmatrix}}$$

We will next consider the system of projective transformations leaving invariant a point and two lines through this point. We will take the invariant point as the origin and the two invariant lines as the axes of coordinates. This condition is brought about by putting $A_2'' = A_1'' = A'' = 0$ in equations (XIV).



If we do this we get

$$\left.\begin{array}{ll}
(1) & K_2 = KK_1 \\
(2) & 0 = 0 \quad \text{an identity} \\
(3) & 0 = 0 \quad \text{an identity} \\
(4) & \dfrac{K_2 - 1}{A_2'} = \dfrac{K(K_1 - 1)}{A_1'} + \dfrac{K-1}{A'} \\
(5) & \dfrac{K_2' - 1}{B_2''} = \dfrac{K'(K_1' - 1)}{B_1''} + \dfrac{K'-1}{B''} \\
(6) & K_2' = K'K_1'
\end{array}\right\} \quad (XIX)$$

The four parameters here are K, K', A', B'', or the two independent cross ratios along the sides of the invariant triangle and the x coordinate of one vertex and the y coordinate of the other. K_2, K_2', A_2', B_2'' of the aufeinanderfolge are completely determined by the four equations (XIX).

The equations of a collineation of this system may be written

$$x_1 = \frac{Kx}{\frac{K-1}{A'}x + \frac{K'-1}{B''}y + 1}$$

$$y_1 = \frac{K'y}{\frac{K-1}{A'}x + \frac{K'-1}{B''}y + 1} \qquad (XX)$$

A second collineation of the same system is

$$x_2 = \frac{K_1 x}{\frac{K_1-1}{A_1'}x + \frac{K_1'-1}{B_1''}y + 1}$$

$$y_2 = \frac{K_1' y}{\frac{K_1-1}{A_1'}x + \frac{K_1'-1}{B_1''}y + 1} \qquad (XXI)$$

The aufeinander folge may be written

$$X_2 = \frac{KK_1X}{\left(\frac{K(K_1-1)}{A_1'} + \frac{K-1}{A'}\right)X + \left(\frac{K(K_1'-1)}{B_1''} + \frac{K'-1}{B''}\right)y + 1}$$

(XXII)

$$y_2 = \frac{K'K_1'}{\left(\frac{K(K_1-1)}{A_1'} + \frac{K-1}{A'}\right)X + \left(\frac{K(K_1'-1)}{B_1''} + \frac{K'-1}{B''}\right)y + 1}$$

and is of same form as (XX)
The inverse of XX obtained as before
may be written

$$X = \frac{\frac{1}{K}x_1}{\frac{1-K}{KA'}X_1 + \frac{1}{K'B''}y_1 + 1}$$

(XXIII)

$$y = \frac{\frac{1}{K_1}y_1}{\frac{1-K}{KA'}X_1 + \frac{1}{K'B''}y_1 + 1}$$

This is also of the same form as (XX) and will leave invariant the given point and the two given lines. Thus the system under discussion possesses the group properties.
∴ There is a four parameter

group leaving invariant a given point and two given lines through the point. The parameters are A', B'', $K K'$. This is the group $G_4(ll')$.

The determinant form of the equation of a collineation of $G_4(ll')$ may be written

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & KA' \\ 0 & B'' & 1 & 0 \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ 0 & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & 0 \\ 0 & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ 0 & 0 & 1 & 0 \\ A' & 0 & 1 & K \\ 0 & B'' & 1 & K' \end{vmatrix}} \qquad (XXIV)$$

Next consider the system of collineations leaving invariant two points and two lines, the two points being on one of the lines and the two lines through one of the points. We can bring about this condition by putting $A_2' = A_1' = A_1$ in equations (XIX). Thus we get

(1) $K_2 = KK_1$

(2) $0 = 0$ an identity

(3) $0 = 0$ an identity (XXV)

(4) $\dfrac{(K_2-1)}{A'} = \dfrac{K(K_1-1)}{A'} + \dfrac{K-1}{A'}$ or $K_2 = KK_1$

(5) $\dfrac{K_2'-1}{B_2''} = \dfrac{K'(K_1'-1)}{B_1''} + \dfrac{K'-1}{B''}$

(6) $K_2' = K'K_1'$

(4) is the same as (1) ∴ the number of equations is reduced to ~~three~~ these three uniquely determine K_2, K_2', B_2'', of the aufeinanderfolge.

The equation of a collineation of the system may be written

$$x_1 = \frac{Kx}{\frac{K-1}{A'}x + \frac{K'-1}{B''}y + 1}$$

$$(\overline{XXVI})$$

$$y_1 = \frac{K'y}{\frac{K-1}{A'}x + \frac{K'-1}{B''}y + 1}$$

A second collineation is

$$x_2 = \frac{K_1 x_1}{\frac{K_1-1}{A_1'}x + \frac{K_1'-1}{B_1''}y + 1}$$

$$(\overline{XXVII})$$

$$y_2 = \frac{K_1'y}{\frac{K_1-1}{A'}x + \frac{K_1'-1}{B_1''}y + 1}$$

The aufeinander folge is

$$x_2 = \frac{KK_1x}{\frac{KK_1-1}{A'}x + \left(\frac{K'(K_1'-1)}{B_1''} + \frac{K'-1}{B''}\right)y + 1}$$

$$(\overline{XXVIII})$$

$$y_2 = \frac{K'K_1'y}{\frac{KK_1-1}{A'}x + \left(\frac{K'(K_1'-1)}{B_1''} + \frac{K'-1}{B''}\right)y + 1}$$

and is of same form as $\overline{XXVI}$

The inverse may be written

$$x = \frac{\frac{1}{K} x_1}{\frac{1-K}{KA'} x_1 + \frac{1}{K'B''} y_1 + 1} \qquad \boxed{XXIX}$$

$$y = \frac{\frac{1}{K'} y_1}{\frac{1-K}{KA'} x_1 - \frac{K'-1}{K'B''} y_1 + 1}$$

and is of same form as XXVI and will leave the two given lines and the two given points invariant.

[In all the equations of this system A' is not a parameter but is a constant of the system]

Thus the system under discussion possesses all the fundamental group properties.

∴ There is a three parameter group leaving invariant two given points and two given lines two of the points being on one of the lines and two of the lines being through one of the points.

The parameters of the group are $KK'B''$. This is the group $G_3(ABl')$

The determinant form of a collineation of this group is the same as that of $G_4(All')$ A' being a constant for this group.*

Let us next study the system of collineations leaving invariant three lines. To do this we put in equations (XXV) $B_2'' = B_1'' = B''$. We get

(1) $K_2 = KK_1$

(2) $0 = 0$ an identity

(3) $0 = 0$ an identity (XXX)

(4) $\frac{K_2 - 1}{A'} = \frac{K(K_1 - 1)}{A'} + \frac{K-1}{A'}$ or $K_2 = KK_1$

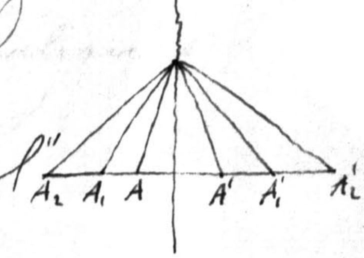
(5) $\frac{K_2' - 1}{B_1''} = \frac{K'(K_1' - 1)}{B''} + \frac{K' - 1}{B''}$ or $K_2' = K'K_1'$

(6) $K_2' = K'K_1'$

We have left only two independent equations. These determine uniquely the parameters k_2 and k_2' of the aufeinanderfolge. The equations of the collineations of this system, aufeinanderfolge, and inverse are identical with those of $G_3(ABC')$. However in this case neither A' nor B'' are parameters but are constants of the system. This system then possesses all the fundamental group properties.

$\therefore$ There is a two parameter group leaving invariant three given lines. The two parameters of this group are k and k' the two independent cross ratios of the one dimensional transformations along the sides of the invariant triangle. This is the group $G_2(A.B.C)$

Another system of transformations not in the regular line of development given above is the system leaving invariant a line and a point not on the line. In order to investigate this system of transformations let us take the invariant line as the axis of x, and take the axis of y through the invariant point. This we can accomplish by putting $A_2''=A_1''=A''=0$ and $B_2''=B_1''=B''$ in equations (XI). The equations then reduce to:—



$$(1)\quad \frac{K_2A_2'-A_2}{A_2'-A_2K_2}=\frac{(K_1A_1'-A_1)(KA'-A)+AA'(1-K_1)(K-1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

$$(2)\quad \frac{A_2A_2'(1-K_2)}{A_2'-A_2K_2}=\frac{AA'(K_1A_1'-A_1)(1-K)+A_1A_1'(1-K_1)(A'-AK)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

$$(3)\quad \frac{A_2A_2'(1-K_2)}{A_2'-A_2K_2}=\frac{AA'(K_1A_1'-A_1)(1-K)+A_1A_1'(A'-AK)(1-K_1)}{(A'-AK)(A_1'-A_1K)+AA'(1-K)(K_1-1)} \qquad (\boxed{XXXI})$$

$$(4)\quad \frac{K_2-1}{A_2'-A_2K_2}=\frac{(K_1-1)(KA'-A)+(K-1)(A_1'-A_1K_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

$$(5)\quad \frac{A_2'(1-K_2)+A_2(K_2'-K_2)}{A_2'-A_2K_2}=\frac{AA'(K_1-1)(1-K)+(A_1'-A_1K_1)(A'-KA)+K'K_1'(A'-A)(A_1-A_1')}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

$$(6)\quad \frac{K_2'(A_2'-A_2)}{A_2'-A_2K_2}=\frac{K'K_1'(A'-A)(A_1'-A_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}$$

(2) is exactly identical with (3) and if we change signs in (6) throughout and add one to both sides we get (5). Thus among the six equations there are two relations and the number of equations really reduces to (4). Thus there are just enough equations to determine uniquely the four parameters of the aufeinanderfolge A_2, A_2', K_2, K_2'.

The equations of a collineation of this system may be written

$$x_1 = \frac{\dfrac{KA'-A}{A'-AK}x + \dfrac{AA'(1-K)}{A'-AK}y + \dfrac{AA'(1-K)}{A'-AK}}{\dfrac{K-1}{A'-AK}x + \dfrac{A'(1-K)+A(K'-K)}{A'-AK} + 1} \qquad \text{(XXXII)}$$

$$y_1 = \frac{\dfrac{K'(A'-A)}{A'-AK}y}{\dfrac{K-1}{A'-AK}x + \dfrac{A'(1-K)+A(K'-K)}{A'-AK} + 1}$$

Another may be written

$$x_2 = \frac{\dfrac{K_1A_1'-A_1}{A_1'-A_1K_1}x_1 + \dfrac{A_1A_1'(1-K_1)}{A_1'-A_1K_1}y_1 + \dfrac{A_1A_1'(1-K_1)}{A_1'-A_1K_1}}{\dfrac{K_1-1}{A_1'-A_1K_1}x_1 + \dfrac{A_1'(1-K_1)+A(K_1'-K_1)}{A_1'-A_1K_1} + 1}$$

$$\boxed{\mathrm{XXXIII}}$$

$$y_2 = \frac{\dfrac{K_1'(A_1'-A_1)}{A_1'-A_1K_1}\,y_1}{\dfrac{K_1-1}{A_1'-A_1K_1}\,x_1 + \dfrac{A_1'(1-K_1)+A(K_1'-K_1)}{A_1'-A_1K_1}\,y_1 + 1}$$

The aufeinander folge may be written as follows and is seen to be of the same form

$$y_2 = \frac{\dfrac{(K_1A_1'-A_1)(KA'-A)+AA'(1-K_1)(K-1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}\,x + \dfrac{AA'(K_1A_1'-A_1)(1-K)+A_1A_1'(A'-AK)(1-K_1)}{(A'-AK)(A_1'-A_1K)+AA'(1-K)(K_1-1)}\,y + \dfrac{AA'(K_1A_1'-A_1)(1-K)+A_1A_1'(A'-AK)(1-K_1)}{(A'-AK)(A_1'-A_1K)+AA'(1-K)(K_1-1)}}{\dfrac{(K_1-1)(KA'-A)+(K-1)(A_1'-A_1K_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}\,x + \dfrac{AA'(K_1-1)(1-K)+(A_1'-A_1K_1)(A'-KA)+K'K_1'(A'-A)(A_1-A_1')}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}\,y + 1}$$

$$\boxed{\mathrm{XXXIV}}$$

$$y_2 = \frac{\dfrac{K'K_1'(A'-A)(A_1'-A_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}\,y}{\dfrac{(K_1-1)(KA'-A)+(K-1)(A_1'-A_1K_1)}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}\,x + \dfrac{AA'(K_1-1)(1-K)+(A_1'-A_1K_1)(A'-KA)+K'K_1'(A'-A)(A_1-A_1')}{(A'-AK)(A_1'-A_1K_1)+AA'(1-K)(K_1-1)}\,y + 1}$$

The inverse of $\overline{\mathrm{XXXII}}$ may easily be obtained by substituting $A''=0$ in $(\boxed{\mathrm{XIII}})$. If we do this we get

$$x = \frac{\dfrac{A'-AK}{KA'-A}x_1 - \dfrac{AA'[K'(A-A')+(A'-AK)](1-K)-[A'-AK]AA'(1-K)}{K'B''(KA'-A)(A'-A)}y_1 - \dfrac{AA'(1-K)}{KA'-A}}{\dfrac{1-K}{KA'-A}x_1 - \dfrac{AA'(K-1)(1-K)-(KA'-A)[K'(A-A')+(A'-AK)]}{K'B''(KA'-A)(A'-A)}y_1 + 1}$$

$$\boxed{XXXV}.$$

$$y = \frac{\dfrac{(A'-AK)(KA'-A)-(K-1)AA'(1-K)}{(KA'-A)K'(A'-A)}y_1}{\dfrac{1-K}{KA'-A}x_1 - \dfrac{AA'(K-1)(1-K)-(KA'-A)[K'(A-A')+(A'-AK)]}{K'B''(KA'-A)(A'-A)}y_1 + 1}.$$

These Equations are of the same form as $\boxed{XXXII}$.

Thus the system under discussion possesses the fundamental group properties.

∴ There is a four parameter group having invariant a line and a point not on the line. The four parameters are A, A', K, K' or two of the x coordinates of the vertices of the invariant triangle and the two independent cross-ratios of the one dimensional transformations along the sides of the invariant triangle. This is the group $G_4(Al'')$. A collineation of this group may be written in determinant form as follows.

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & 0 & 1 & A \\ A' & 0 & 1 & KA' \\ 0 & B'' & 1 & 0 \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ 0 & B'' & 1 & K' \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & 0 & 1 & 0 \\ A' & 0 & 1 & 0 \\ 0 & B'' & 1 & K'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & 0 & 1 & 1 \\ A' & 0 & 1 & K \\ 0 & B'' & 1 & K' \end{vmatrix}}$$